calc III notes 12/3/2021

Stokes' Theorem: if $S$ is a nice surface w/ a really nice boundary and $\vec{F}$ is a v.f. on $\mathbb{R}^3$ w/ components having cts partial derivatives on $S$, then

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

NB ① $\operatorname{curl}(\vec{F})$ is sometimes nicer than $\vec{F}$

② sometimes the line integral is easier than the surface integral

③ if $S$ and $T$ are surfaces w/ $\partial S = \partial T$

then $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_T \operatorname{curl}(\vec{F}) \cdot d\vec{S}$

when $S \cup T$ does not enclose a discontinuity of $\operatorname{curl}(\vec{F})$

EX: compute $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$ for $\vec{F} = \langle xz, yz, xy \rangle$ and $S$ is the part of sphere $x^2+y^2+z^2=4$ inside the cylinder $x^2+y^2=1$ above the xy plane

Solution 1: compute directly

sorry for this awful picture!

parameterize $S$ via $\vec{S}(r,\theta) = \langle r\cos\theta, r\sin\theta, \underbrace{\sqrt{4-r^2}} \rangle$

$\downarrow$

$x^2+y^2+z^2=4$

$z = \pm\sqrt{4-x^2-y^2}$

since $z \geq 0$

$z = \sqrt{4-r^2}$

on $[r,\theta] \in [0,1] \times [0, 2\pi]$

from the cylinder

$\vec{S}_r = \langle \cos\theta, \sin\theta, \frac{1}{2}(4-r^2)^{-1/2}(-2r) = \langle \cos\theta, \sin\theta, -r(4-r^2)^{-1/2} \rangle$

$\vec{S}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\vec{S}_r \times \vec{S}_\theta = \det\begin{bmatrix} i & j & K \\ \cos\theta & \sin\theta & -r(4-r^2)^{-1/2} \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix}$$

$= i(r^2\cos\theta(4-r^2)^{-1/2}) - j(-r^2\sin\theta(4-r^2)^{-1/2}) + K(r\cos^2\theta + r\sin^2\theta)$

$= \langle r^2\cos\theta(4-r^2)^{-1/2}, r^2\sin\theta(4-r^2)^{-1/2}, r \rangle$

$$\text{curl}(\vec{F}) = \text{"}\nabla \times \vec{F}\text{"} = \det\begin{bmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & yz & xy \end{bmatrix}$$

$= i\left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(yz)\right) - j\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(xz)\right) + K\left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xz)\right)$

$= \langle x-y, x-y, 0 \rangle$

$\text{curl}(\vec{F})(\vec{S}(r,\theta)) = \langle r\cos\theta - r\sin\theta, r\cos\theta - r\sin\theta, 0 \rangle$

$\text{curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta) = (r\cos\theta - r\sin\theta)\langle 1,1,0 \rangle \cdot r\langle r\cos\theta(4-r^2)^{-1/2}, r\sin\theta(4-r^2)^{-1/2}, 1 \rangle$

$= (r^2\cos\theta - r^2\sin\theta)(r\cos\theta(4-r^2)^{-1/2} + r\sin\theta(4-r^2)^{-1/2})$

$= r^3\cos^2\theta(4-r^2)^{-1/2} - r^3\sin^2\theta(4-r^2)^{-1/2}$

$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$= r^3(4-r^2)^{-1/2}\cos(2\theta)$

$\iint_S \text{curl}(\vec{F})\,d\vec{S} = \iint_D \text{curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta)\,dA = \iint_D r^3(4-r^2)^{-1/2}\cos(2\theta)\,dA$

$\int_0^{2\pi} \cos(2\theta) \int_0^1 r^3(4-r^2)^{-1/2}\,dr\,d\theta$ $\quad 4-r^2 = w \quad r^2 = 4-w$

$-2r\,dr = dw$

$r\,dr = -\frac{1}{2}dw$

$-\frac{1}{2}\int_0^{2\pi} \cos(2\theta) \int (4-w)(w)^{-1/2}\,dw\,d\theta$

$4-w = u \quad w^{-1/2}dw = dv$

$du = -dw \quad v = 2w^{1/2}$

$-\frac{1}{2}\int_0^{2\pi} \cos(2\theta)\left((4-w)(2w^{1/2}) + \int 2w^{1/2}\,dw\right)d\theta$

$-\frac{1}{2}\int_0^{2\pi} \cos(2\theta)\left[(4-w)(2w^{1/2}) + \frac{4}{3}w^{3/2}\right]d\theta$

$$-\frac{1}{2}\int_0^{2\pi}\cos(2\theta)\left[(4-(4-r^2))(2(4-r^2))^{1/2}+\frac{4}{3}(4-r^2)^{3/2}\right]_0^1 d\theta$$

$$-\frac{1}{2}\int_0^{2\pi}\cos(2\theta)\left[r^2(8-2r^2)^{1/2}+\frac{4}{3}(4-r^2)^{3/2}\right]_0^1 d\theta$$

$$-\frac{1}{2}\int_0^{2\pi}\cos(2\theta)\left[\sqrt{6}+4\sqrt{3}-\frac{32}{3}\right]d\theta$$

$$-\frac{1}{2}\left[\frac{3\sqrt{6}+12\sqrt{3}-32}{3}\right]\int_0^{2\pi}\cos(2\theta)\,d\theta$$

$2\theta=\gamma \qquad 2d\theta=d\gamma \qquad d\theta=1/2\,d\gamma$

$$-\left[\frac{3\sqrt{6}+12\sqrt{3}-32}{6}\right]\int_0^{2\pi}\frac{1}{2}\cos(\gamma)\,d\gamma$$

$$-\left[\frac{3\sqrt{6}+12\sqrt{3}-32}{12}\right]\sin(2\theta)\Big|_0^{2\pi}$$

0 at $2\pi$ and 0

so $\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}=0$

Solution 2: using Stokes' theorem

parameterize $\partial S$ via $\vec{r}(\theta)=\langle\cos\theta,\sin\theta,\sqrt{3}\rangle$

from $x^2+y^2+z^2=4 \qquad x^2+y^2=1$

$1+z^2=4$

$z^2=3$

$z=\sqrt{3} \qquad (z>0)$

$\theta\in[0,2\pi]$

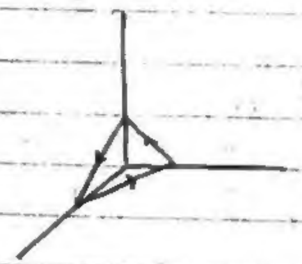
$r'(\theta)=\langle-\sin\theta,\cos\theta,0\rangle$

$\vec{F}(r(\theta))=\langle\sqrt{3}\cos\theta,\sqrt{3}\sin\theta,\sin\theta\cos\theta\rangle$

$\vec{F}(r(\theta))\cdot r'(\theta)=-\sqrt{3}\sin\theta\cos\theta+\sqrt{3}\sin\theta\cos\theta=0$

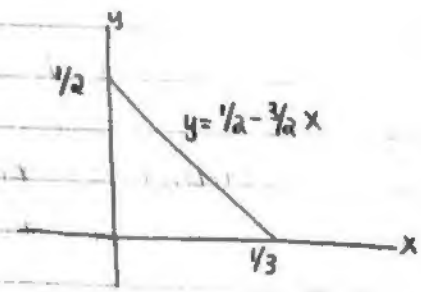
$\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}=\int_{\partial S}\vec{F}\cdot d\vec{r}=\int_0^{2\pi}\vec{F}(r(\theta))\cdot r'(\theta)\,d\theta=\int_0^{2\pi}0\,d\theta=0$

EX: compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle 1, x+yz, xy-\sqrt{z} \rangle$ on $C$ the intersection of the plane $3x+2y+z=1$ w/ coordinate planes in the first octant oriented counterclockwise from above.



use stockes' theorem b/c the curve has 3 pieces / is piecewise defined in 3 cases ie gross

$C = \partial S$ parametrize $S$: SHADOW




from $3x+2y+z=1$

$3x=1$ $\quad 2y=1$
$x=1/3$ $\quad y=1/2$

$$\vec{S}(x,y) = \langle x, y, 1-3x-2y \rangle \text{ on } D = \{(x,y) : 0 \le x \le 1/3,\ 0 \le y \le 1/2 - 3/2 x\}$$

$$\text{curl}(\vec{F}) = \text{"}\nabla \times F\text{"} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix} = \langle x-y, -y, 1 \rangle$$

$$\text{curl}(\vec{F})(\vec{S}(x,y)) = \langle x-y, -y, 1 \rangle$$

$$\vec{S}_x = \langle 1, 0, -3 \rangle \quad \vec{S}_y = \langle 0, 1, -2 \rangle$$

$$\vec{S}_x \times \vec{S}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = \langle 3, 2, 1 \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_D \text{curl}(\vec{F})\vec{S}(x,y) \cdot (\vec{S}_x \times \vec{S}_y)\, dA$$

$$\iint_D \langle x-y, -y, 1\rangle \cdot \langle 3,2,1\rangle \, dA$$

$$\iint_D 3x-3y-2y+1 \, dA = \iint_D 3x-5y+1 \, dA$$

$$\int_0^{1/3}\int_0^{1/2-3/2x} 3x-5y+1 \, dy \, dx$$

$$\int_0^{1/3}\left[3xy-\frac{5}{2}y^2+y\right]_0^{1/2-3/2x} dx$$

$$\int_0^{1/3}\left[3x(1/2-3/2x)-\frac{5}{2}(1/2-3/2x)^2+(1/2-3/2x)\right)dx$$

$$\int_0^{1/3}\left(\frac{3}{2}x-\frac{9}{2}x^2-\frac{5}{2}\left(\frac{1}{4}+\frac{9}{4}x^2-\frac{3}{2}x\right)+\frac{1}{2}-\frac{3}{2}x \quad dx$$

$$\int_0^{1/3}\left(-\frac{9}{2}x^2-\frac{5}{8}-\frac{45}{8}x^2+\frac{15}{4}x+\frac{1}{2} \, dx$$

$$\int_0^{1/3} -\frac{81}{8}x^2+\frac{15}{4}x-\frac{1}{8} \quad dx$$

$$\left.-\frac{81}{24}x^3+\frac{15}{8}x^2-\frac{1}{8}x\right|_0^{1/3}$$

$$-\frac{81}{24}\frac{1}{27}+\frac{15}{8}\frac{1}{9}-\frac{1}{8}\frac{1}{3} =$$
$$-3/24+5/24-1/24 = \boxed{1/24}$$

EXERCISE: $\int_C \vec{F}\cdot d\vec{r}$ $\quad \vec{F}=\langle 2y, xz, x+y\rangle$

C is the curve of the intersection of the plane

$z=y+2$ and the cylinder $x^2+y^2=4$